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NOTES ON

THE THEORY OF HEAVE ATTENUATION

by

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This note examines the power required for heave attenuation of SEV, using either cushion air dumping or flow modulating fans. Shortage of time has prevented a complete analysis of some aspects, particularly for the modulated flow case which is of most practical interest. Power requirements have been calculated for 100% heave attenuation only in the latter case and the values obtained are naturally very large. More work needs to be done on the partial alleviation case, which is of more practical interest.

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ABSTRACT

This note examines the power required for heave attenuation of SEV, using either cushion air dumping or flow modulating fans. Shortage of time has prevented a complete analysis of some aspects, particularly for the modulated flow case which is of most practical interest. Power requirements have been calculated for 100% heave attenuation only in the latter case and the values obtained are naturally very large. More work needs to be done on the partial alleviation case, which is of more practical interest.

The paper first establishes the cushion volume change in a sinusoidal sea and from this calculates the instantaneous pressure. The so-called "compressibility terms" increase in size by a factor $L^{3/2}$ (where L is the cushion length) when Froude scaling, but this is due to speed, not size. They are also dependent on the fan characteristics, and vanish if $\partial p/\partial Q = 0$. They do not exist, therefore, in the case of 100% heave attenuation by fan modulation. Even when they do exist, it is not clear that they necessarily <u>increase</u> the dynamic cushion pressure excursions.

Partly for this reason, and partly because compressibility terms complicate the analysis, the heave motion is then solved for the incompressible case only. Simple equations are obtained for the power dissipated in heave attentuation.

The power requirement of a fan flow modulation Heave Attenuation System (HAS) appears very sensitive to the total head losses which occur between the fan and the cushion, and to vehicle speed.

It is believed that this note establishes the feasibility of studying HAS dynamics analytically and thus establishing broad trends and identifying potentially optimum solutions. But there have been insufficient time available to go all the way to these final objectives in this first look at the problem.

INTRODUCTION

The purpose of this paper is to investigate the approximate air supply requirements of an ACV/SES in regular waves, in a simple yet realistic manner. One of the first such analyses was by Beardsley[1], who considered only the "platforming" case over sinusoidal waves. Subsequent analyses have included many variables so that solutions can only be obtained numerically. It is the intent of this present paper to bridge the gap, to a certain extent, between the rather extreme simplifications of Beardsley and the more sophisticated computer studies, without losing the virtues of simplicity and closed form solutions.

Since our purpose is to obtain approximate "order of magnitude" results, we first note (Figure 1) that the parameter

$$p_o Q_o = (W/S) Q_o \, \alpha \, W$$

or

$$\frac{p_o Q_o}{W} = \frac{Q_o}{S} = \text{constant, approximately}$$

Here p_o and Q_o are the equilibrium values of the cushion pressure and air volume flow. As shown in Figure 1, most vehicles fall in the range

$$4 > \frac{p_o Q_o}{W} > \frac{1}{4} \qquad \text{(ft lb/sec)}$$

$p_o Q_o$ is the measure of the energy lost in the leaking cushion air. Thus, a well-sealed SES tends to have low values; ACV's with a large clearance gap have high values.

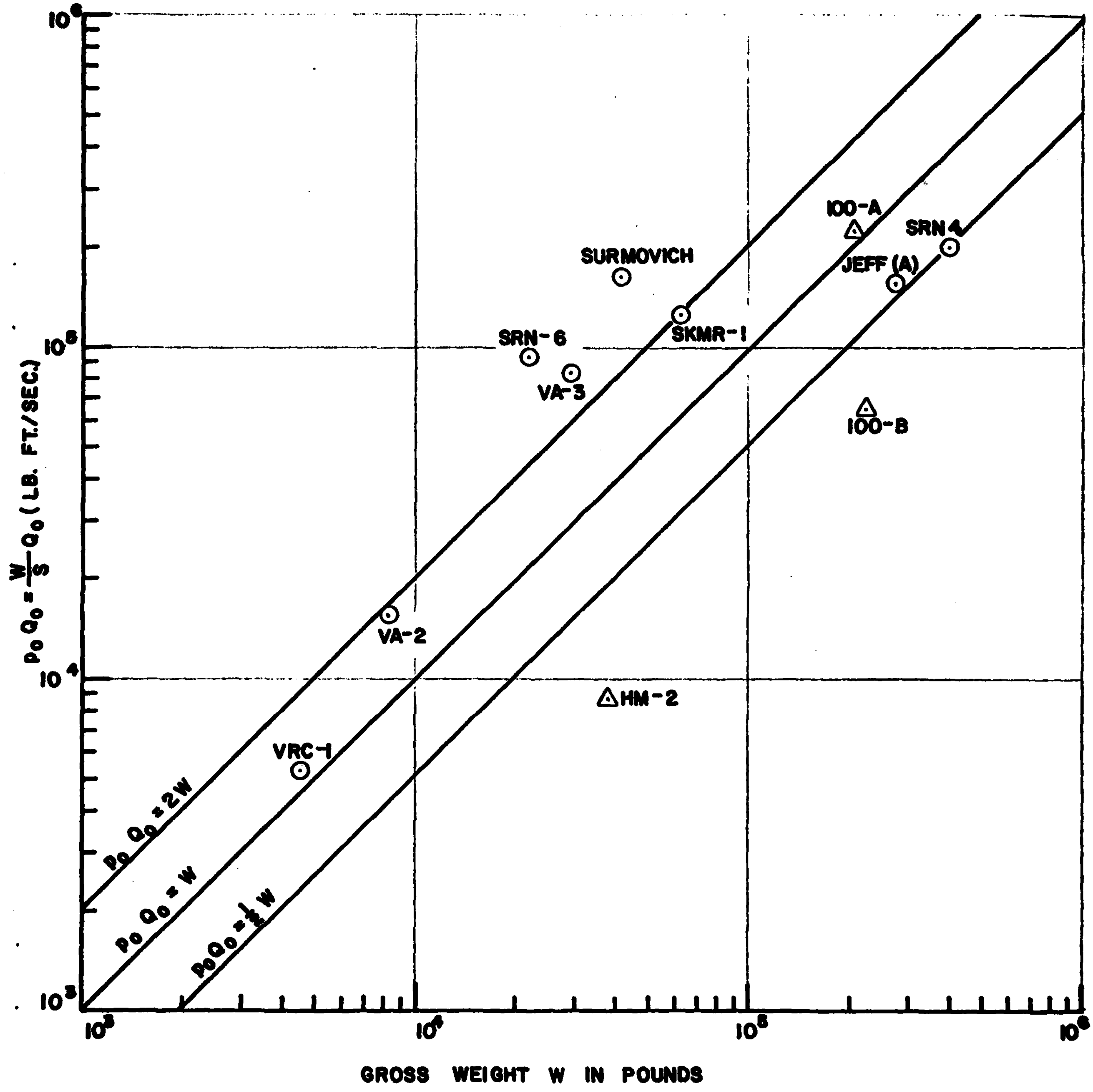

Figure 1. Approximate Variation of the Parameter $p_o Q_o$ (= cushion pressure x volume flow rate) with Vehicle Gross Weight.

THE CUSHION VOLUME

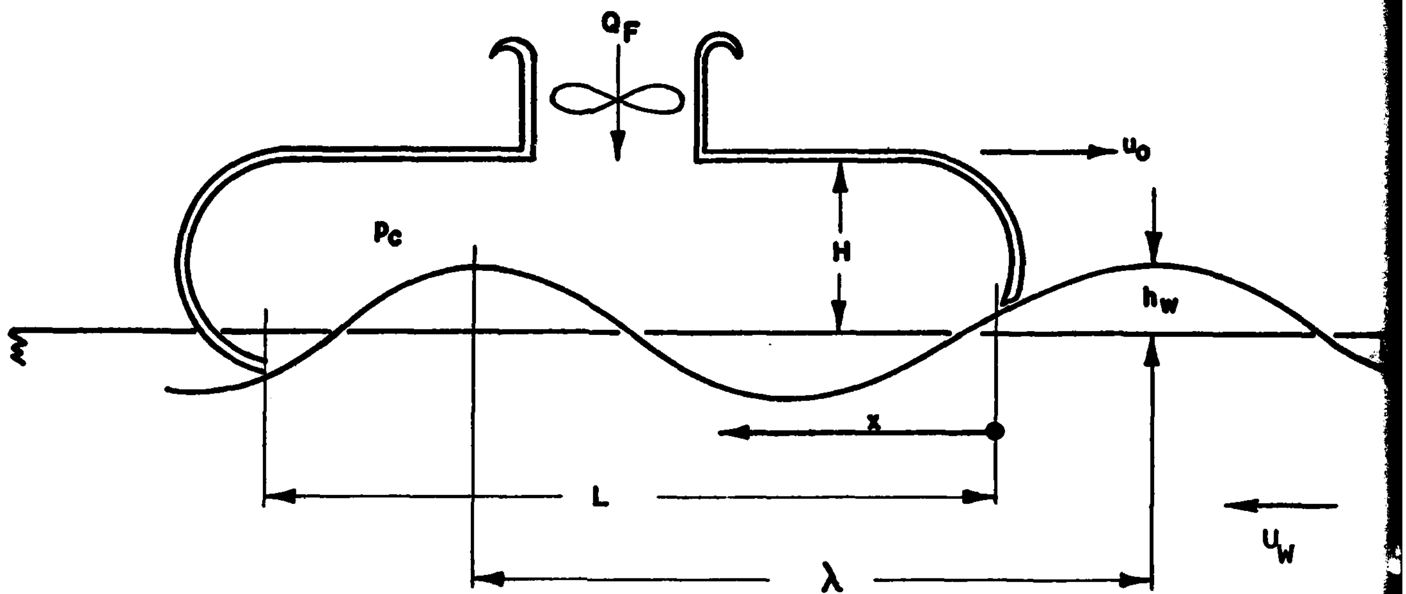

Figure 2. Basic Geometry.

We assume a craft, at forward speed u_o, heading into a sinusoidal sea. The craft planform is rectangular, of length L and beam B, and is assumed to be incapable ot pitching. The "air gap" is negligible in relation to the cushion height H. The local wave elevation is

$$h = h_w \sin 2\pi\left[\frac{x}{\lambda} + \frac{(u_o+u_w)}{\lambda} t\right] \tag{1}$$

At a given time t, the cushion volume will be

$$V = LBH - h_w B\lambda \int_o^{L/\lambda} \sin 2\pi \left[\frac{x}{\lambda} + \frac{(u_o+u_w)}{\lambda} t\right] d(x/\lambda) \tag{2}$$

$$= LBH - \frac{h_w B\lambda}{2\pi} \left\{\cos 2\pi\left[\frac{L}{\lambda} + \frac{(u_o+u_w)}{\lambda} t\right] - \cos 2\pi \frac{(u_o+u_w)}{\lambda} t\right\}$$

$$= LBH - \frac{h_w B\lambda}{2\pi} \sqrt{\sin^2 2\pi(L/\lambda) + [\cos 2\pi(L/\lambda) - 1]^2} \sin\left[\frac{2\pi(U)}{\lambda} t +$$

(where ϕ_1 is a phase angle, and is not important, and $U = u_o + u_w$, the speed relative to the waves.) Continuing the redution:

$$V = LBH - \frac{h_w B\lambda}{2\pi}\sqrt{2[1 - \cos 2\pi(L/\lambda)]}\ \sin[2\pi\ (U/\lambda)\ t + \phi_1] \tag{3}$$

We pause here to note that, for comparison with simple "piston theory," the displacement (δ) of the "wave piston" is given by

$$\frac{\delta}{h_w} = \frac{\sqrt{2[1-\cos\ 2\pi(L/\lambda)]}}{2\pi(L/\lambda)}\ \sin\left[2\pi(U/\lambda)\ t + \phi_1\right]$$

$$= \frac{\sin\ \pi(L/\lambda)}{\pi(L/\lambda)}\ \sin\left[2\pi(U/\lambda)\ t + \phi_1\right] \tag{4}$$

This is the result given (for example) by Mantle.[2]

Returning to equation (3), the rate of change of cushion volume is

$$\frac{dV}{dt} = LB\left\{\frac{dH}{dt} - h_w\left[\frac{\sin\ \pi(L/\lambda)}{\pi(L/\lambda)}\right]2\pi\ (U/\lambda)\ \cos\left[2\pi(U/\lambda)\ t + \phi_1\right]\right\}$$

$$\doteq LB\left[\frac{dH}{dt} - \Omega h_w F(L/\lambda)\ \cos\ (\Omega t + \phi_1)\right] \tag{5}$$

when $\Omega = 2\pi(U/\lambda)$, the frequency of wave encounter in rads/sec.

and $F(L/\lambda) = \dfrac{\sin\ \pi(L/\lambda)}{\pi(L/\lambda)}$

We shall show later that the heave term dH/dt is generally negligible by comparison with the cos Ωt term. The phase angle ϕ_1 may usually be ommitted, since it can be made zero by a suitable selection of zero time.

THE CUSHION PRESSURE

Let

$p_c + p_\infty$ = absolute cushion pressure

$p_o + p_\infty$ = absolute cushion pressure under equilibrium conditions

Q_F = fan volume flow rate

Q_o = fan volume flow rate under equilibrium conditions

m_o = *mass of air in the cushion under equilibrium conditions*

$\dot{m}$ = air mass flow rate into the cushion (sum of flows in and out)

V_o = cushion volume under equilibrium conditions

ρ_o = equilibrium cushion density = m_o/V_o

t = o at an equlibrium conditions

At any time t the density of the cushion air is

$$\rho = \frac{m_o + \int_o^t \dot{m}\, dt}{V} \tag{6}$$

For adiabatic conditions

$$\left(\frac{p_c + p_\infty}{p_o + p_\infty}\right)^{1/\gamma} = \frac{\rho}{\rho_o} = \frac{V_o}{V}\left(1 + \frac{1}{m_o}\int_o^t \dot{m}\, dt\right) \tag{7}$$

$$\therefore \frac{1}{m_o}\int_o^t \dot{m}\, dt = \frac{V}{V_o}\left(\frac{p_c + p_\infty}{p_o + p_\infty}\right)^{1/\gamma} - 1$$

Differentiating with respect to time

$$\frac{\dot{m}}{m_o} = \frac{1}{V_o}\frac{dV}{dt}\left(\frac{p_c + p_\infty}{p_o + p_\infty}\right)^{1/\gamma} + \frac{V}{\gamma V_o}\left(\frac{p_c + p_\infty}{p_o + p_\infty}\right)^{1/\gamma - 1}\left(\frac{1}{p_o + p_\infty}\right)\frac{dp_c}{dt} \tag{8}$$

Since $m_o = \rho_o V_o$

$$\frac{\dot{m}}{\rho_\infty} = \frac{\rho_o}{\rho_\infty}\left(\frac{p_c + p_\infty}{p_o + p_\infty}\right)^{1/\gamma}\left[\frac{dV}{dt} + \frac{V}{\gamma(p_c + p_\infty)}\frac{dp}{dt}\right]$$

$$= \left(\frac{p_c}{p_\infty} + 1\right)^{1/\gamma}\left[\frac{dV}{dt} + \frac{V}{\gamma(p_c+p_\infty)}\ \frac{dp}{dt}\right] \tag{9}$$

$$= \frac{dV}{dt} \quad \text{for incompressible flow } (\gamma \to \infty)$$

If it is assumed that $p_c = p_o + \Delta p$ and that $\Delta p/p_\infty << 1$, then equation (9) becomes

$$\frac{\dot{m}}{p_\infty} = \left(\frac{p_o}{p_\infty} + 1\right)^{\frac{1}{\gamma}}\left[\frac{dV}{dt} + \frac{V}{\gamma(p_o + p_\infty)}\ \frac{dp}{dt}\right] \tag{9a}$$

The flow out of the cushion through the leakage area (a) is

$$Q_j = a\sqrt{(2/\rho)\ p_c} = a\sqrt{(2/\rho_\infty)(p_o+\Delta p)}$$

The flow into the cushion from the fan, (neglecting inertia terms) is

$$Q_F = Q_o + \frac{\partial Q}{\partial p}\,\Delta p$$

Making these substitutions in equation (9)

$$Q_o + \frac{\partial Q}{\partial p}\,\Delta p - a\sqrt{(2/\rho_\infty)(p_o+\Delta p)} = \left(\frac{p_o}{p_\infty} + 1\right)^{1/\gamma}\left[\frac{dV}{dt} + \frac{V}{\gamma}\,\frac{(p_o+p_\infty)^{1/\gamma-1}}{p_\infty^{1/\gamma}}\ \frac{dp}{dt}\right] \tag{10}$$

here V and dV/dt are given by equations (3) and (5). This equation cannot be solved explicitly for Δp, even though it is first order, because of the square root. An approximate solution is discussed in Appendix II.

Some convenient non-dimensionalizations are

$$\zeta = \frac{Q_o}{P_o}\,\frac{\partial p}{\partial Q}\ * \qquad \xi = \frac{1}{Q_o}\,\frac{dV}{dt} \tag{10a}$$

---

* $\partial Q/\partial p$ is not a true partial derivative (as witness its subsequent inversion) but the notation used is the most familiar in the literature. In retrospect the more general expression

$$\Delta p = c_o + c_1 Q_F + c_2 Q_F^2$$

\- might lead to a more general, and thus more useful result.

Equation (10) then has the alternative form

$$1 + \frac{1}{\zeta}\frac{\Delta p}{p_o} - \sqrt{1 + \Delta p/p_o} = \xi(1+p_o/p_\infty)^{1/\gamma} + \left\{\frac{p_o}{p_\infty}(1+p_o/p_\infty)^{1/\gamma-1}\frac{V}{\gamma Q_o}\frac{dp}{dt}\right\} \qquad (11)$$

The terms in the brackets are the so-called "non-scaling" and "compressibility" terms. (For example, see Lavis, et al[3]). The curly bracket is equal to zero for incompressible flow ($\gamma \rightarrow \infty$), the "non-scaling" term equals unity.

If fan modulation is employed to eliminate heave, $dp/dt = 0$, then an incompressible flow analysis is in error by the factor $(p_o/p_\infty + 1)^{1/\gamma}$, i.e.

if	p_o	=	10	50	100	200 lb/ft^2
$(p_o/p_\infty+1)^{1/\gamma}$			1.003	1.017	1.034	1.067

This is small enough to be neglected in an analysis of the present type. Formally, one can account for it by assuming a wave height which is lower by this factor.

On the other hand, without attenuation, we shall find (equations 18 and 18a) that for incompressible flow, approximately

$$\Delta p \approx p_o\left(\frac{2\zeta}{2-\zeta}\right)\xi$$

so that

$$\frac{dp}{dt} \approx p_o\left(\frac{2\zeta}{2-\zeta}\right)\frac{d\xi}{dt}$$

Equation (10) then becomes, approximately

$$1 + \frac{1}{\zeta}\frac{\Delta p}{p_o} - \sqrt{1+\Delta p/p_o} \approx (1+p_o/p_\infty)^{1/\gamma}\left\{\xi + \left(\frac{p_o/p_\infty}{1+p_o/p_\infty}\right)\left(\frac{2\zeta}{2-\zeta}\right)\frac{V}{\gamma Q_o}\frac{d\xi}{dt}\right\}$$

Substituting equations (3) and (5) for V and dV/dt, and assuming that craft heave motion is small compared to wave motion, so that $LB(dH/dt)$ is negligible compared with dV/dt

$$1 + \frac{1}{\zeta}\frac{\Delta p}{p_o} - \sqrt{1+\Delta p/p_o} = G(1+p_o/p_\infty)^{1/\gamma}\Bigg\{-\cos\theta + \frac{G}{\gamma}\left(\frac{p_o/p_\infty}{1+p_o/p_\infty}\right)\left(\frac{2\zeta}{2-\zeta}\right)\left[\frac{H_o}{h_w}\sin\Omega t - F(L/\lambda)\sin^2\Omega t\right]\Bigg\} \tag{12}$$

where

$$G = \frac{LBh_w\Omega}{Q_o} F(L/\lambda) = 2\pi \frac{U}{Q_o} LB \frac{h_w}{\lambda} = 2\pi U \frac{h_w}{\lambda}\left(\frac{W}{Q_o p_o}\right)$$

Several points are apparent from this result:

1. The "compressibility term" gives a phase shift and, even on these simple assumptions, introduces both a constant term and a second harmonic.

2. Since G is independent of scale, there is no significant scale effect for constant p_o/p_∞,

<u>but</u>

3. Since G varies with absolute speed, it does become important when Froude scaling; it increases as $\sqrt{L}$, so comparing a 30th scale model to full size, G is five times as much in the full scale case. And since, in Froude Scaling, cushion-pressure must vary with length, the total variation is as $L^{3/2}$, for a total factor of 164 in the example of a 1/30th scale model.

4. The "compressibility term" will not be important if the fan characteristic parameter ζ is small. It will be zero if $\partial p/\partial Q = 0$. In practical applications, the parameter ζ varies between 0 and -2.

5. Because of complex phase shift effects, the compressibility terms may either attenuate or increase the heave acceleration, depending on the precise values of the parameters. We show in Appendix II that if the heave velocity dH/dt can be considered negligible, compressibility will always <u>attenuate</u> the ride. It's therefore important to understand this effect, and design (so far as other factors allow) for a favorable effect.

6. The phase shift in peak pressure will generally result in an increased heave attenuation system (HAS) power requirement.

In view of these considerations, the work which follows will ignore the "compressibility" term, and consider only incompressible flow.

HEAVE MOTION FOR INCOMPRESSIBLE FLOW

For incompressible flow equation (12) becomes

$$a\sqrt{\frac{2}{\rho}\,(p_o + \Delta p)} = Q_o + \frac{\partial Q}{\partial p}\,\Delta p - \frac{dV}{dt} \tag{13}$$

Again, let

$$\zeta = \frac{Q_o}{p_o}\,\frac{\partial p}{\partial Q}$$

$$\xi = \frac{1}{Q_o}\,\frac{dV}{dt} \tag{14}$$

Then

$$a\sqrt{\frac{2}{\rho}\,p_o\left(1 + \frac{\Delta p}{p_o}\right)} = Q_o + \frac{\Delta p}{p_o}\,\frac{Q_o}{\zeta} - Q_o\xi \tag{15}$$

And since

$$Q_o = a\sqrt{\frac{2p_o}{\rho}}$$

$$\sqrt{1 + \frac{\Delta p}{p_o}} = 1 + \frac{1}{\zeta}\,\frac{\Delta p}{p_o} - \xi \tag{16}$$

Squaring both sides we obtain the quadratic

$$\left(\frac{\Delta p}{\zeta p_o}\right)^2 + [2(1 - \xi) - \zeta]\left(\frac{\Delta p}{\zeta p_o}\right) + (1 - \xi)^2 - 1 = 0 \tag{17}$$

The plus root is taken in this case.

i.e.

$$\frac{\Delta p}{p_o} = \zeta\left[-\left[\left(1 - \frac{\zeta}{2}\right) - \xi\right] + \sqrt{\left(1 - \frac{\zeta}{2}\right)^2 + \zeta\xi}\;\right] \tag{18}$$

$$= \left(\frac{2\zeta}{2 - \zeta}\right) \xi$$

as

$$\xi \to 0 \tag{18a}$$

Equations (18) and (18a) are plotted in Figure 3. Now if M is the vehicle's mass (W/g) and S (=LB) its cushion area the heave equation of motion is

$$M\frac{d^2H}{dt^2} = S(p_o + \Delta p) - W$$

$$\therefore \quad \frac{d^2H}{dt^2} = \left(\frac{S}{M}\right) \Delta p = g \frac{S}{W} \Delta p = g \frac{\Delta p}{p_o} \tag{19}$$

Then from equations (5), (14) and by integrating equation (19) to obtain $\frac{dH}{dt}$

$$\xi = \frac{LB}{Q_o}\left[\frac{g}{p_o}\int \Delta p \, dt - \Omega h_w F\left(\frac{L}{\lambda}\right)\cos \Omega t\right]$$

$$= \frac{K_H}{p_o}\int \Delta p \, dt - K_w \cos \Omega t \tag{20}$$

where

$$K_H = \frac{Sg}{Q_o}$$

$$K_w = \frac{S\Omega h_w F\left(\frac{L}{\lambda}\right)}{Q_o}$$

Substituting equation (20) for ξ into (16)

$$\sqrt{1 + \frac{\Delta p}{p_o}} = 1 + \frac{1}{\zeta}\frac{\Delta p}{p_o} - \frac{K_H}{p_o}\int \Delta p \, dt + K_w \cos \Omega t \tag{21}$$

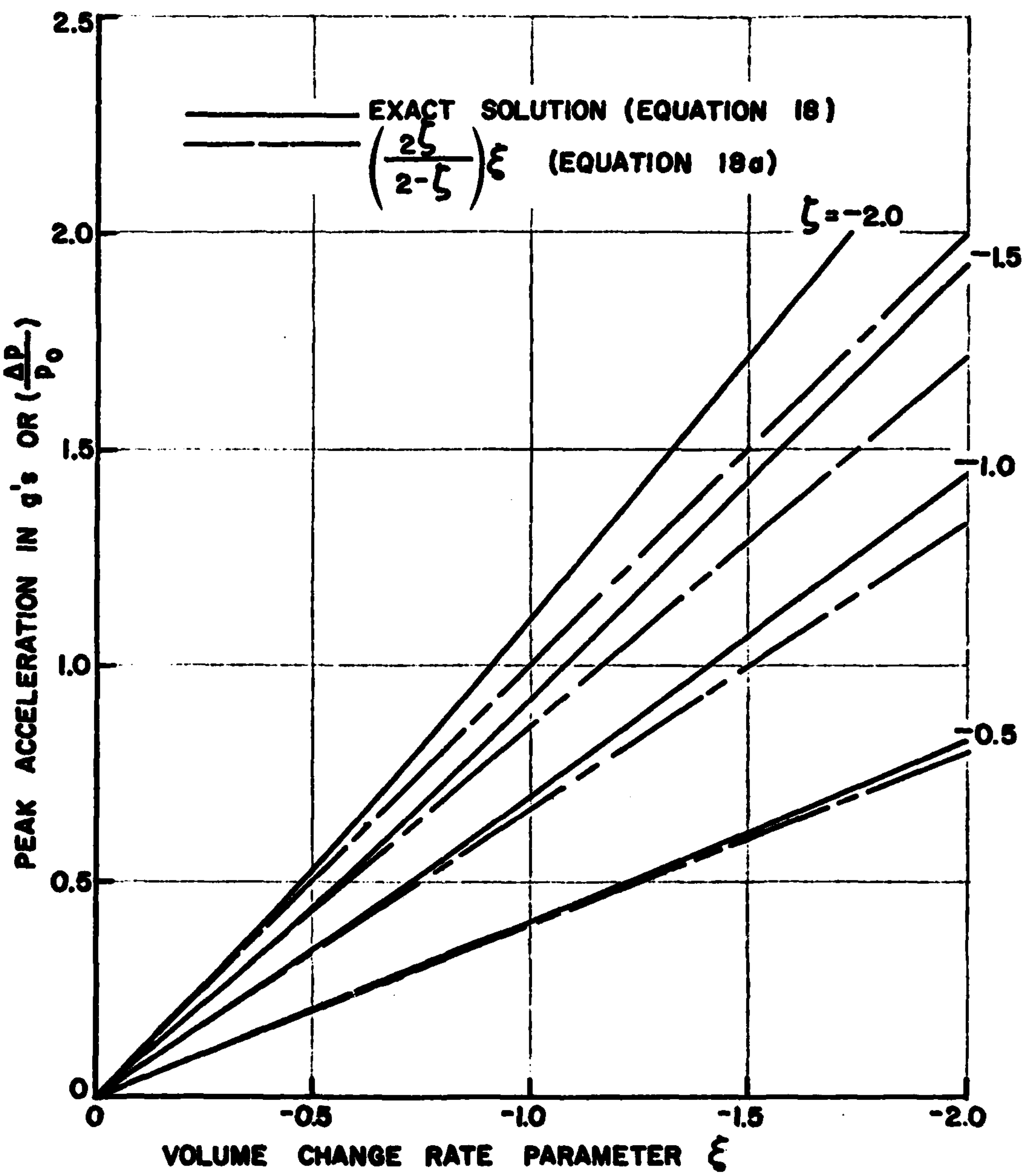

Figure 3. Exact and Approximate Solutions to the Equation for Cushion Pressure. $\Delta p/p_o = f(\xi)$ so $\Delta p/p_o = f(\xi_{max})$ (Equations 18 and 18a).

Rearranging

$$\frac{K_H}{p_o}\int \Delta p \, dt = 1 + \frac{1}{\zeta}\frac{\Delta p}{p_o} + K_w \cos \Omega t - \sqrt{1 + \frac{\Delta p}{p_o}}$$

Differentiating with respect to time

$$\frac{K_H}{p_o}\Delta p = \frac{1}{\zeta p_o}\frac{dp}{dt} - \Omega K_w \sin \Omega t - \frac{1}{2p_o\sqrt{1 + \frac{\Delta p}{p_o}}}\frac{dp}{dt}$$

$$\therefore \quad \left[\frac{1}{\zeta} - \frac{1}{2\sqrt{1 + \frac{\Delta p}{p_o}}}\right]\frac{dp}{dt} - K_H \Delta p = p_o \Omega K_w \sin \Omega t \tag{22}$$

or

$$\left.\begin{aligned} &\frac{dp}{dt} - Z_1 \Delta p = Z_2 \sin \Omega t \\ &\text{where} \\ &Z_1 = \frac{K_H}{\left[\frac{1}{\zeta} - \frac{1}{2\sqrt{1 + \frac{\Delta p}{p_o}}}\right]} \\ &Z_2 = \frac{p_o \Omega K_w}{\left[\frac{1}{\zeta} - \frac{1}{2\sqrt{1 + \frac{\Delta p}{p_o}}}\right]} \end{aligned}\right\} \tag{23}$$

If we formally restrict the analysis to $\Delta p \ll p_o$

$$Z_1 \approx \frac{2\zeta K_H}{[2-\zeta]} = \frac{S_q}{Q_o}\,\frac{2\zeta}{2-\zeta} = g\,\frac{W}{p_o Q_o}\left(\frac{2\zeta}{2-\zeta}\right)$$

$$Z_2 \approx \frac{2\zeta p_o \Omega K_w}{[2-\zeta]} = \frac{W h_w \Omega^2}{Q_o}\,F\left(\frac{L}{\lambda}\right)\left(\frac{2\zeta}{2-\zeta}\right)$$

and

$$\Delta p \approx e^{Z_1 t}\left[\int e^{-Z_1 t} Z_2 \sin \Omega t\, dt + C\right]$$

$$= -\frac{Z_2}{\Omega}\cos \Omega t + \Delta p_o e^{Z_1 t} \qquad (24)$$

Thus for steady-state conditions (no transients)

$$\frac{\Delta p}{p_o} = -\left(\frac{S h_w \Omega}{Q_o}\right) F\left(\frac{L}{\lambda}\right)\left(\frac{2\zeta}{[2-\zeta]}\right)\cos \Omega t \qquad (25)$$

So finally, noting (19) and comparing equation (25) with the equivalent "piston motion" of equation (4) we get the rather simple result

$$\frac{\ddot{H}}{\ddot{\delta}} = \frac{\text{craft acceleration}}{\text{piston acceleration}} = \left(\frac{gS}{\Omega Q_o}\right)\frac{2\zeta}{[2-\zeta]} \qquad (26)$$

Comparing (25) with (18a) we see that they are identical*, if the dH/dt term in equation (5) is dropped. So, to a first order, we may neglect the heave motion of the vehicle in computing the air flow into and out of the cushion.

---

* From (18a) and (5)

$$\frac{\Delta p}{p_o} = \xi\left(\frac{2\zeta}{2-\zeta}\right) = \frac{S}{Q_o}\,\frac{dH}{dt} - \frac{S h_w \Omega}{Q_o}\,F\left(\frac{L}{\lambda}\right)\left(\frac{2\zeta}{2-\zeta}\right)\cos \Omega t$$

The transient term in equation (24) is independent of the forcing term Z_2, and if $h_w = 0$, $Z_2 = 0$ and

$$\Delta p = \Delta p_o e^{Z_1 t} \tag{27}$$

where

$$Z_1 = \left(\frac{Sg}{Q_o}\right)\frac{2\zeta}{[2 - \zeta]}$$

Thus the motion will be unstable if $0 < \zeta < +2$; a known result.* For all other values $Z_1 < 0$ and the transient decays, so that the motion is stable.

<u>Heave Attenuation by Dumping</u>

Let $\theta = \Omega t$ and the suffixes o and D refer to dump valve shut and open. Δp_1 is the pressure differential when the dump valve opens (at $\Omega t = \theta_1$) and Δp_2 the valve when it closes, as indicated in Figure 4.

By definition

$$Q_o = a\sqrt{\frac{2}{\rho} P_o} \qquad \text{[Equation (16)]}$$

Also

$$Q_D = a_D\sqrt{\frac{2}{\rho} P_o}$$

where a_D is the normal leakage area <u>plus</u> the dump valve area.

$$\left.\begin{aligned} z_o &= \left(\frac{Sg}{Q_o}\right)\left(\frac{2\zeta}{2 - \zeta}\right)_o \\ z_D &= \left(\frac{Sg}{Q_D}\right)\left(\frac{2\zeta}{2 - \zeta}\right)_D \end{aligned}\right\} \quad \text{[corresponding to } Z_1 \text{ in equation (23)]}$$

* A result first noticed by Walker[4,5]

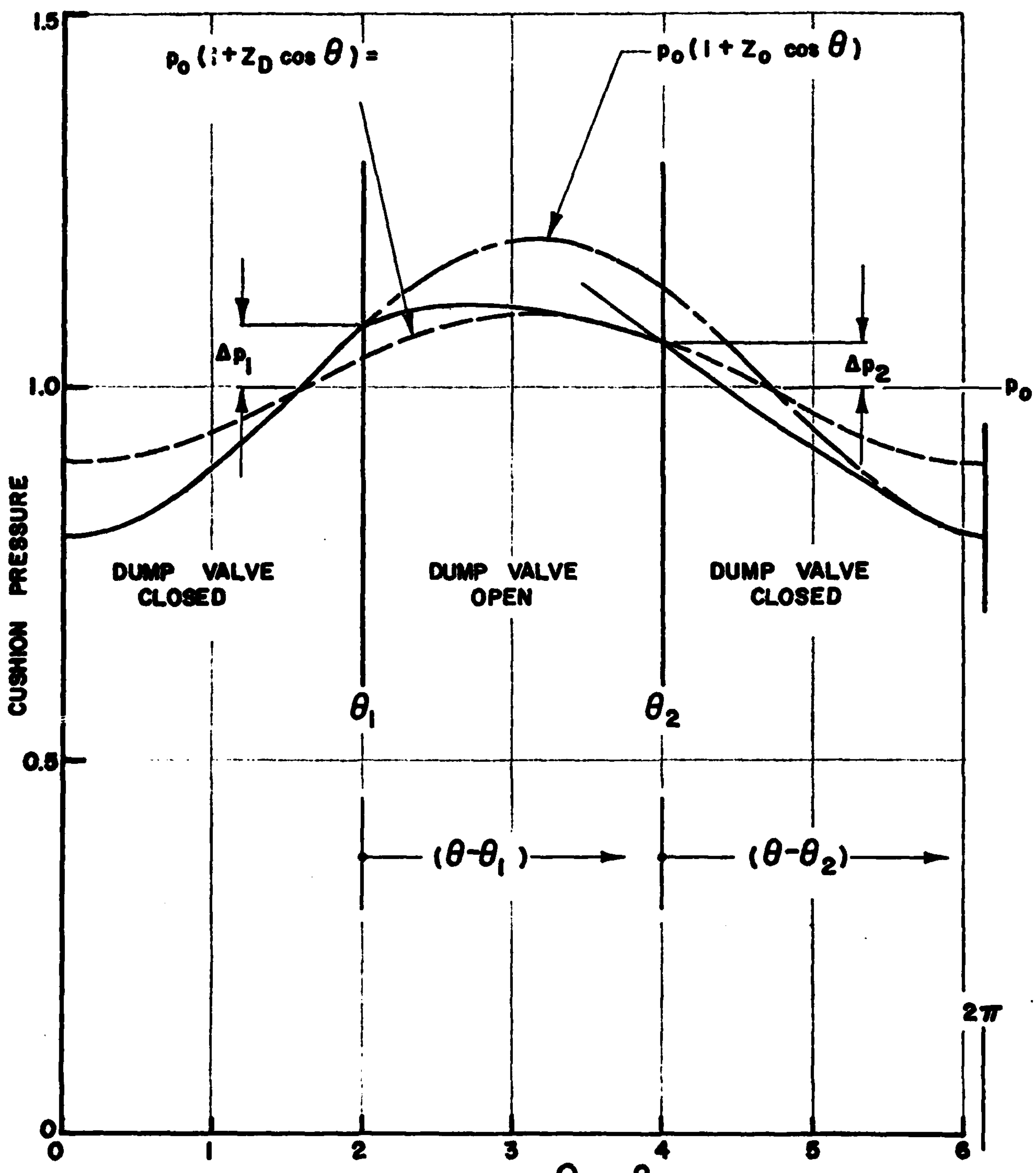

Figure 4. Variation of Cushion Pressure with Active Ride Control.

$$Z_o = \frac{Sh_w\Omega}{Q_o} f\left(\frac{L}{\lambda}\right)\left(\frac{2\zeta}{2-\zeta}\right)_o$$

$$Z_D = \frac{Sh_w\Omega}{Q_D} f\left(\frac{L}{\lambda}\right)\left(\frac{2\zeta}{2-\zeta}\right)_D$$

[corresponding to $Z_2/p_o\Omega$ in equation (23)] (28)

when the dump valve is closed

$$\Delta p = -p_o Z_o \cos\theta + \Delta p_2 e^{z_o \Delta t_2}$$

and open

$$\Delta p = -p_o Z_D \cos\theta + \Delta p_1 e^{z_D \Delta t_1}$$

[corresponding to equation (24)] (29)

$$\Delta t_2 = \frac{\theta - \theta_2}{\Omega}$$

$$\Delta t_1 = \frac{\theta - \theta_1}{\Omega}$$

So at the point of opening the dump valve, at θ_1

$$\Delta p_1 = -p_o Z_o \cos\theta_1 + \Delta p_2\, e^{[z_o/\Omega(\theta_1 - \theta_2)]} \tag{30a}$$

at the point of closing the dump valve, at θ_2

$$\Delta p_2 = -p_o Z_D \cos\theta_2 + \Delta p_1 e^{[z_D/\Omega\,(\theta_2 - \theta_1)]} \tag{30b}$$

By substituting in (30a) for Δp_2 from (30b)

$$\Delta p_1 = -p_o Z_o \cos\theta_1 + \left[-p_o Z_D \cos\theta_2 + \Delta p_1 e^{\frac{Z_D}{\Omega}(\theta_2 - \theta_1)}\right] e^{\frac{Z_o}{\Omega}(\theta_1 - \theta_2)}$$

$$\Delta p_1 \left[1 - e^{\frac{Z_o}{\Omega}\left(\frac{Z_D}{Z_0} - 1\right)(\theta_2-\theta_1)}\right] = -p_o \left[Z_o \cos\theta_1 + Z_D \cos\theta_2 e^{\frac{Z_o}{\Omega}(\theta_1-\theta_2)}\right]$$

$$\therefore \quad \left(\frac{\Delta p_1}{p_o}\right) = Z_o \frac{\left[\cos\theta_1 + \frac{Z_D}{Z_o}\cos\theta_2\, e^{\frac{Z_o}{\Omega}(\theta_1-\theta_2)}\right]}{\left[e^{\frac{Z_o}{\Omega}\left(\frac{Z_D}{Z_o} - 1\right)(\theta_2-\theta_1)} - 1\right]} \tag{31}$$

Similarly, substituting for Δp_1 in (30b)

$$\therefore \quad \frac{\Delta p_2}{p_o} = Z_o \frac{\left[\cos\theta_1 + \frac{Z_D}{Z_o}\cos\theta_2\, e^{\frac{Z_D}{\Omega}(\theta_2-\theta_1)}\right]}{\left[e^{\frac{Z_o}{\Omega}\left(1-\frac{Z_D}{Z_o}\right)(\theta_1-\theta_2)} - 1\right]} \tag{32}$$

So by selecting values for θ_1, θ_2 and Z_D/Z_o we can compute Δp_1 and Δp_2, and hence evaluate equations (29). Because of the exponential terms we cannot easily obtain maxima analytically. The portion of the solution in which Δp_{max} will occur is, from (29) and (31)

$$\frac{\Delta p}{p_o} = Z_o \left\{\frac{Z_D}{Z_o}\cos\theta + \left[\frac{\cos\theta_1 + \frac{Z_D}{Z_o}\cos\theta_2\, e^{\frac{Z_o}{\Omega}(\theta_1-\theta_2)}}{e^{\frac{Z_o}{\Omega}\left(\frac{Z_D}{Z_o} - 1\right)(\theta_2-\theta_1)} - 1}\right] e^{\frac{Z_D}{\Omega}(\theta-\theta_1)}\right\} \tag{33}$$

It's clear that the maxima must be sought graphically or iteratively; which is easy to do by computer, of course. Note that the variables are four in number i.e. $$\frac{\Delta p}{p_o} = f\,(\theta_1\ \theta_2\ Z_o\ Z_D)$$

A Solution in Which the Transient Terms are Ignored

In this case, when the dump valve is open [from equation (29)]

$$\frac{\Delta p}{p_o} = Z_D \cos\theta \qquad (\theta_1 < \theta < \theta_2) \qquad (34)$$

$$\left(\frac{\Delta p}{p_o}\right)_{max} = -Z_D$$

And the attenuation ratio is $$\frac{\left(\Delta p_{max}\right)_D}{\left(\Delta p_{max}\right)_o} = \frac{Z_D}{Z_o}$$

We open the dump valve when Δp is equal to the maximum value of Δp which will occur with the dump valve open.

i.e. $$\left(\frac{\Delta p}{p_o}\right)_{max} = \frac{\Delta p_1}{p_o} = Z_o \cos\theta_1 = -Z_D$$

$$\therefore \quad \cos\theta_1 = \frac{-Z_D}{Z_o}$$

$$\sin\theta_1 = \sqrt{1 - \left(\frac{Z_D}{Z_o}\right)^2}$$

Also, from Figure 4

$$\theta_2 = 2\pi - \theta_1$$

so that

$$\sin\theta_2 = -\sin\theta_1$$

The instantaneous power lost through the dump valve is

$$P_D = \tfrac{1}{2}\,\dot{m}_D\,\frac{2}{\rho}(p_o + \Delta p) \simeq \tfrac{1}{2}\,\rho\,(a_D - a)\left(\frac{2}{\rho}\,p_o\right)^{3/2}\left(1 + \frac{3}{2}\,\frac{\Delta p}{p_o}\right)$$

$$= \tfrac{1}{2}\rho\,(a_D - a)\left(\frac{2}{\rho}\,p_o\right)^{3/2}\left(1 + \frac{3}{2}\,Z_D\cos\theta\right) \quad \text{[from equation 34]} \qquad (35)$$

Averaged over one cycle this is, as a ratio of equilibrium power

$$\frac{P_{DAV}}{P_o} = \left(\frac{a_D}{a} - 1\right) \frac{1}{2\pi} [(\theta_2 - \theta_1) + \frac{3}{2} Z_D(\sin \theta_2 - \sin \theta_1)]$$

$$= \left(\frac{a_D}{a} - 1\right) \left[\left(\frac{\pi - \theta_1}{\pi}\right) - \frac{3}{2\pi} Z_D \sqrt{1 - \left(\frac{Z_D}{Z_o}\right)^2}\right] \tag{36}$$

Thus the power ratio is not a simple function of area ratio, a_D/a_o, but depends upon the attenuation ratio:

$$\frac{Z_D}{Z_o} \left(= \frac{\text{peak acceleration with HAS}}{\text{peak acceleration without HAS}}\right)^{*}$$

and $$\frac{a_D}{a} \left(= \frac{\text{leakage and dump area}}{\text{leakage area}}\right)$$

and $$Z_D = \frac{S}{a_D} \frac{h_w \Omega}{\sqrt{\frac{2}{\rho}}\sqrt{\frac{W}{S}}} F\left(\frac{L}{\lambda}\right)\left(\frac{2\zeta}{2 - \zeta}\right)_D = -\left(\frac{\Delta p}{p_o}\right)_{max}$$

There is therefore no simple way to present this result unless we assume $\zeta_D = \zeta_o$, so that

$$\frac{a_D}{a} = \frac{Z_o}{Z_D} \tag{37}$$

Then

$$\frac{P_{DAV}}{P_o} = \frac{\Delta_1 P}{P_o} + \frac{\Delta_2 P}{P_o} \tag{38}$$

---

*HAS = Heave Attenuation System.

- and only the second term depends upon the details of the ship and waves. These two terms are plotted in Figure 5. The second term may generally be neglected, except under very rough conditions, when large accelerations are experienced with the HAS activated.

Attenuation with a Variable Supply Fan

Heave attenuation can also be provided by a variable pitch fan, or any one of several other ways of modulating the fan's flow. In this section, we assume that the modulation is loss-free.

The flow required from the fan is

$$Q = \frac{dV}{dt} + a\sqrt{\frac{2}{\rho}\left(p_o + \Delta p\right)} \qquad \text{(a version of equation 13)}$$

$$= \frac{dV}{dt} + Q_o\sqrt{1 + \frac{\Delta p}{p_o}} \qquad (39)$$

The total head rise through the fan itself is

$$\Delta H = p_o\left(1 + \frac{\Delta p}{p_o}\right) + (1 - \eta_T)\ \tfrac{1}{2}\ \rho\left(\frac{Q}{A}\right)^2 \qquad (40)$$

where $(1 - \eta_T)$ is the total head loss divided by the dynamic head.

Let η_F = the fan efficiency $\frac{\Delta HQ}{P}$ (≈ 0.8)

η_L = the total lift system efficiency ($= p_o Q_o / P \approx 0.4$)

$$\frac{\eta_L}{\eta_F} = \frac{p_o Q_o}{P} \bigg/ \frac{\Delta H_o Q_o}{P} = \frac{\Delta H_o - \left(1 - \eta_T\right)\ \tfrac{1}{2}\ \rho\left(\frac{Q_o}{A}\right)^2}{\Delta H_o} \qquad \text{(from equation 40)}$$

$$\therefore \quad \left(1 - \eta_T\right)\ \tfrac{1}{2}\ \rho\left(\frac{Q_o}{A}\right)^2 = \Delta H_o\left(1 - \frac{\eta_L}{\eta_F}\right)$$

$$= \left[p_o + \left(1 - \eta_T\right)\tfrac{1}{2}\ \rho\left(\frac{Q_o}{A}\right)^2\right]\left(1 - \frac{\eta_L}{\eta_F}\right)$$

Power Increment Required for Heave Attenuation

$$\frac{\Delta P}{P_o} = \frac{\text{Power Increment}}{\text{Calm Water Power}} = \frac{\Delta_1 P}{P_o} + \frac{\Delta_2 P}{P_o}$$

G_p = Peak Acceleration (in g's) with HAS active.

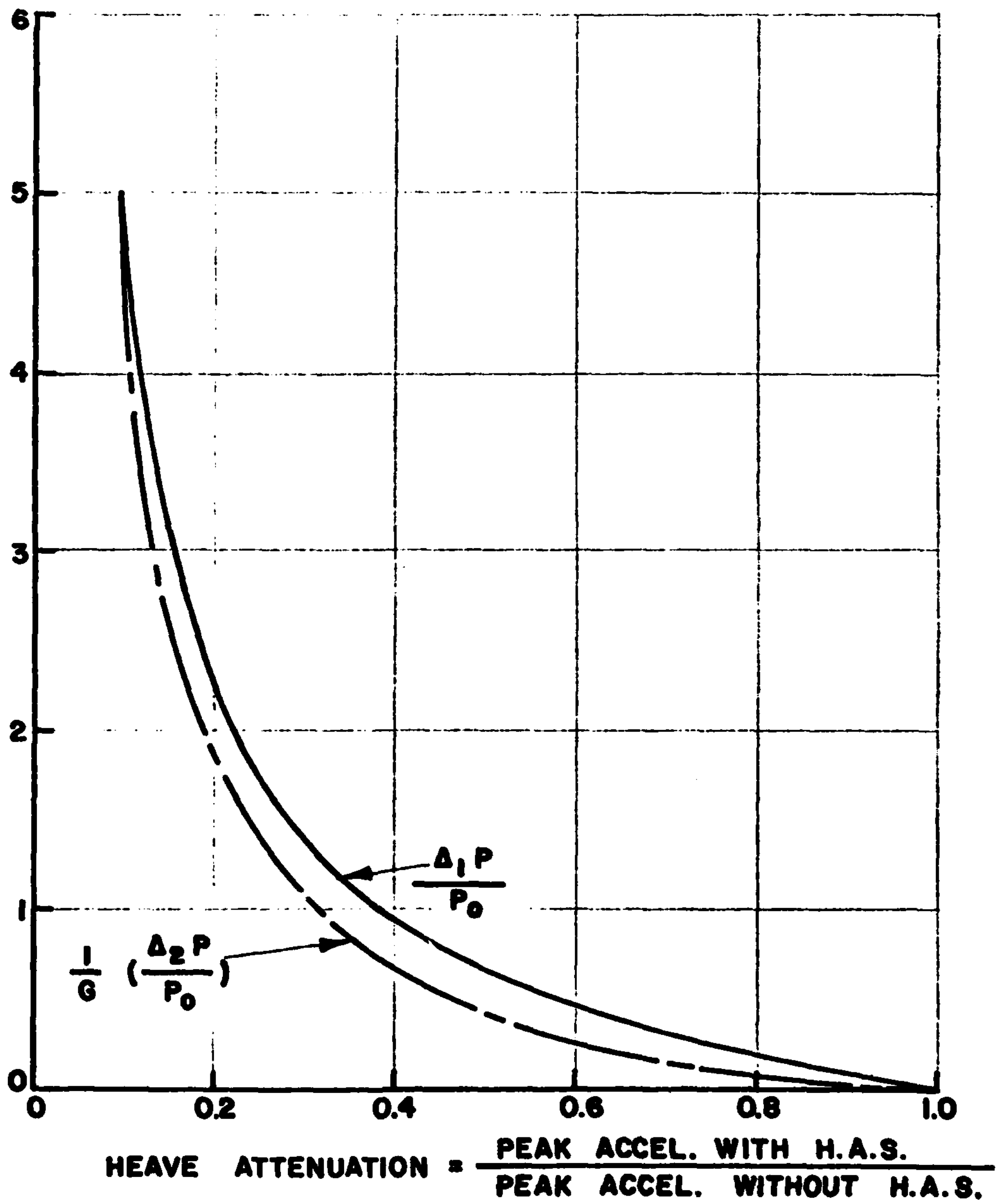

Figure 5. Power Required for Heave Attenuation by Dumping. (The Δ_2 parameter must be multiplied by G_p before substitution in the equation for $\Delta P/P_o$.)

$$\therefore \quad (1 - \eta_T)\, \tfrac{1}{2}\rho \left(\frac{Q_o}{A}\right)^2 = p_o \left(\frac{\eta_F}{\eta_L} - 1\right) = \S p_o \quad \text{say} \tag{41}$$

Reverting to equation (40) the instantaneous power required is given by

$$\eta_F P = \Delta HQ = \left[p_o \left(1 + \frac{\Delta p}{p_o}\right) + \S p_o \left(\frac{Q}{Q_o}\right)^2\right] \left[\frac{dV}{dt} + Q_o \left(1 + \frac{\Delta p}{p_o}\right)\right] \tag{42}$$

Let $\xi = \frac{1}{Q_o} \frac{dV}{dt}$ as before

$$q = 1 + \frac{\Delta p}{p_o} \tag{42a}$$

$$\hat{p} = \left(\frac{\Delta p}{p_o}\right)_{max}$$

Then

$$\frac{Q}{Q_o} = (\xi + q) \qquad \text{(from equation 39)}$$

$$\begin{aligned} \frac{\eta_F P}{p_o Q_o} &= [q + \S(\xi + q)^2](\xi + q) \\ &= q(\xi + q) + \S(\xi + q)^3 \end{aligned} \tag{43}$$

This implies that when $\Delta H > 0$ and $Q < 0$ (or visa-versa) the fan abstracts power from the air like a windmill, which may not be realistic unless a flywheel is used. Note from (39) and (40) that although Q can be negative, ΔH cannot. Thus the equation (40) relationship for ΔH requires modification when $Q < 0$.

Now for sinusoidal waves we have, from (20)

$$\xi = \frac{1}{Q_o} \frac{dV}{dt} = K_w \cos \Omega t = K_w \cos \theta \tag{44}$$

where

$$K_w = \frac{S\Omega h_w F\left(\frac{L}{\lambda}\right)}{Q_o} \quad \text{as before.} \tag{44a}$$

and from equations (25), (42a) and (36a)

$$q = 1 + \frac{\Delta p}{p_o} = 1 - \hat{p} \cos \Omega t = 1 - \hat{p} \cos \theta \tag{45}$$

Substituting for q and ξ in equation (43)

$$\therefore \quad \frac{\eta_F \; P}{p_o Q_o} = (1 - \hat{p} \cos \theta)[1 + (K_w - \hat{p}) \cos \theta] + \xi[1 + (K_w - \hat{p}) \cos \theta]^3$$

$$= 1 - \hat{p} \cos \theta + (K_w - \hat{p}) \cos \theta - \hat{p} \, (K_w - \hat{p})^3 \cos^2\theta$$

$$+ \xi[1 + 3(K_w - \hat{p}) \cos \theta + 3(K_w - \hat{p})^2 \cos^2\theta + (K_w - \hat{p}) \cos^3\theta] \tag{46}$$

The average value of the power parameter $\eta_F P/p_o Q_o$, will be

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta$$

and only the constants and the $\cos^2\theta$ terms contribute

$$\frac{\eta_F P_{AV}}{p_o Q_o} = (1 + \xi) - \tfrac{1}{2}\xi\hat{p}(K_w - \hat{p}) + \frac{3}{2} \xi \, (K_w - \hat{p})^2 \tag{47}$$

= (Equilibrium Value) + increase due to waves and heave attenuatio

For complete alleviation of heave

$$\hat{p} = (\Delta p/p_o)_{max} = 0$$

and $$\left(\frac{\eta_F P_{AV}}{p_o Q_o}\right)_{\hat{p}=0} = \frac{\eta_F}{\eta_L} + \frac{3}{2} K_w^{\,2} \left(\frac{\eta_F}{\eta_L} - 1\right) \tag{48}$$

and $$\frac{\Delta P_{HAS}}{P_o} = \frac{\text{Power to Eliminate Heave}}{\text{Equilibrium Power}} = \frac{3}{2} K_w^{\,2} \left(1 - \frac{\eta_F}{\eta_L}\right)$$

$$= \frac{3}{2}\left(1 - \frac{\eta_L}{\eta_F}\right)\left[\frac{S\Omega h_w}{Q_o} F\left(\frac{L}{\lambda}\right)\right]^2$$

$$= \frac{3}{2}\left(1 - \frac{\eta_L}{\eta_F}\right)\left[2\pi \frac{h_w}{\lambda} F\left(\frac{L}{\lambda}\right)\left(\frac{W}{p_o Q_o}\right) U\right]^2 \tag{49}$$

Note that if $\eta_L = \eta_F$, no power would be required.* The more efficient the lift system duct and diffusion system, the less the penalty.

<u>Without</u> heave attenuation the maximum acceleration would be (from 25)

$$\frac{\Delta p_{max}}{p_o} = \hat{p} = Z_o = \frac{S\Omega h_w}{Q_o} F\left(\frac{L}{\lambda}\right)\left(\frac{2\zeta}{2 - \zeta}\right)$$

$$= K_w f(\zeta) \quad \text{say} \tag{50}$$

So if z is the heave attenuation ratio

$$\hat{p} = zK_w f(\zeta)$$

making this substitution in (47) putting $\$ = (\eta_F/\eta_L - 1)$ (equation 40) and rearranging gives

$$\frac{\eta_F P_{AV}}{p_o Q_o} = \frac{\eta_F}{\eta_L} + \frac{3}{2}\left(\frac{\eta_F}{\eta_L} - 1\right) K_w^{\,2} \left[1 - zf(\zeta)\right]^2 - \tfrac{1}{2} K_w^{\,2} zf(\zeta)\left[1 - zf(\zeta)\right] \tag{51}$$

---

* Strictly true if $\eta_F = 1.0$ or if there is no reverse flow through the fan. When $\eta_F < 1.0$ and reverse flow occurs, the fan absorbs too much energy, by a factor of $1/\eta_F^{\,2}$ in this analysis.

As $\eta_F P_{av}/p_o Q_o$ = equilibrium value (= η_F/η_L) plus the increment $\Delta P_{HAS}/p_o Q_o$, we see that

$$\frac{\Delta P_{HAS}}{P_o} = \tfrac{1}{2} K_W^2 \left\{ 3\left(1 - \frac{\eta_L}{\eta_F}\right) [1 - zf(\zeta)]^2 - \frac{\eta_L}{\eta_F} [1 - zf(\zeta)] \right\} \tag{52}$$

This is finite even for no attenuation (z = 1).

If z = 1 and $\eta_L = \eta_F$

$$\frac{\Delta P_{waves}}{P_o} = - \tfrac{1}{2} K_W^2 [1 - f(\zeta)] \tag{52a}$$

- implying a reduction of power in waves if $f(\zeta) < 1$, which it usually is. But for practical cases, there is an increase in fan power due to waves if (from equation 52)

$$3\left(1 - \frac{\eta_L}{\eta_F}\right)[1 - zf(\zeta)] > \frac{\eta_L}{\eta_F}$$

$$1 - zf(\zeta) > \frac{1}{3\left(\frac{\eta_F}{\eta_L} - 1\right)}$$

$$\left(\approx > \frac{1}{3} \text{ say}\right) \tag{53}$$

- which in general will always be true.

For the purpose of obtaining a rough estimate we consider only the case of complete alleviation, so that, from equations (44a) and (48), the additional power required is

$$\Delta P_{HAS} = \frac{3}{2} \frac{P_o Q_o}{\eta_F} \left(\frac{\eta_F}{\eta_L} - 1\right) \left[\frac{S\Omega h_w}{Q_o} F\left(\frac{L}{\lambda}\right)\right]^2 \quad \text{(lb ft/sec)} \tag{54}$$

$$\frac{\Delta P_{HAS}}{W} = \frac{3}{2} \frac{\left(\frac{\eta_F}{\eta_L} - 1\right)}{\eta_F} \left(\frac{W}{P_o Q_o}\right)\left[\frac{2\pi U h_w}{\lambda} F\left(\frac{L}{\lambda}\right)\right]^2 \quad \text{(ft/sec)} \tag{55}$$

since $\Omega = 2\pi U/\lambda$

This is plotted in Figure 6 for typical values. It's clear that complete suppression of heave is much too expensive of power at high speed on the assumptions used in this analysis. We should examine the implications of equation (52) more closely to see what the trade-offs are for partial alleviation.
is very severe.

For comparison, simple (zero leakage, loss-free dumping) piston theory gives, for 100% alleviation

$$\frac{\Delta P_{HAS}}{W} = \frac{2}{\eta_L} \frac{h_w}{\lambda} F \left(\frac{L}{\lambda}\right) U \qquad \text{(ft/sec)} \qquad (56)$$

Beardsley's[1] wave pumping theory gives

$$\frac{\Delta P_{HAS}}{W} = \frac{2\pi}{\eta_L} \frac{h_w}{\lambda} F \left(\frac{L}{\lambda}\right) U \qquad \text{(ft/sec)} \qquad (57)$$

Both of these give much lower power estimates than equation (55) for typical values. Yet on the other hand, equation (55) predicts no power penalty at all if $\eta_F = \eta_L$. That is, for zero duct and diffusion loss; as noted earlier This is because our equations simulate "energy recovery" by the fan when the flow is negative; also, equations (56) and (57) do not allow for duct losses.

It would seem that minimizing duct and diffusion losses are of paramount importance to maximizing the efficiency of any fan modulation, heave attenuation scheme.

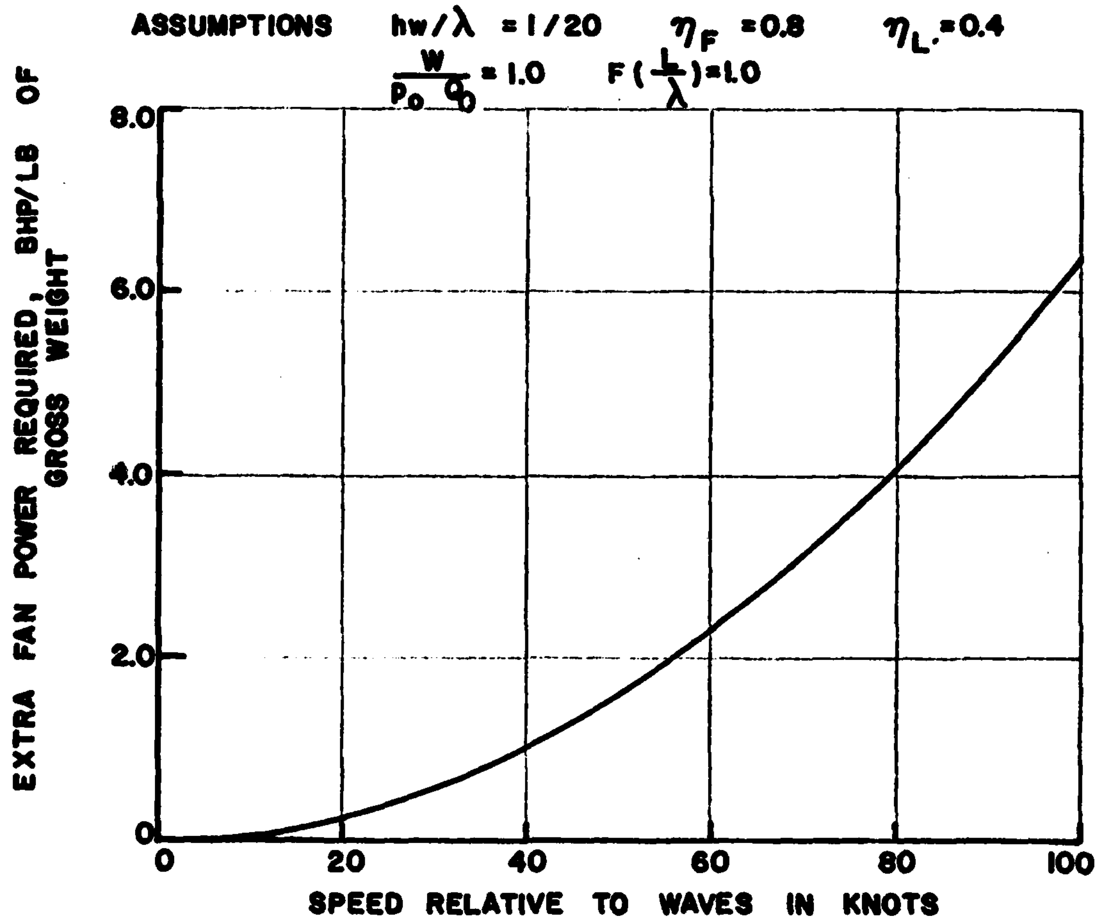

Figure 6. Extra Fan Power Required for Complete Suppression of Heave in Long Waves. In Short Waves, Multiply Power by $[F(L/\lambda)]^2$.

REFERENCES

1. Beardsley, M.W. "Wave Pumping and Its Effects on the Design and Operation of Air-Cushion Ships.," MEL Tech Memo 48/67; USN Marine Engineering Laboratory (March 1967)

2. Mantle, Peter J. A Technical Summary of Air Cushion Craft Development., DTNSRDC Report 4727 (October 1975)

3. Lavis, D.R., and Bartholomew, R.J. "Response of Air Cushion Vehicles to Random Seaways and the Inherent Distortion in Scale Models." Journal of Hydronautics, Vol. 8, No. 3 (July 1974).

4. Walker, N.K., and Shaffer, D.A. "Heave Stability and Heave Damping of Ground Effect Machines - Thick Annular Jet and Plenum Types." USAAVLABS TR 65-26 (July 1965).

5. Walker, N.K. "The Influence of Fan and Ducting Characteristics on the Stability and Performance of Ground Effect Machines." Paper presented at the AIAA General Aviation Design and Operations Meeting (May 1964).

APPENDIX I

FAN CHARACTERISTICS

It's usual to express the total pressure rise (ΔH) across a fan as

$$\psi = \frac{\Delta H}{\rho \omega^2 D^2}$$

and the nondimensional flow rate as

$$\phi = \frac{Q}{\omega D^3}$$

Thus

$$\frac{\partial \psi}{\partial \Delta H} = \frac{1}{\rho \omega^2 D^2}$$

$$\frac{\partial \phi}{\partial Q} = \frac{1}{\omega D^3}$$

$$\frac{\partial \Delta H}{\partial Q} = \frac{\partial \Delta H}{\partial \psi} \frac{\partial \phi}{\partial Q} \frac{\gamma \psi}{\partial \phi} = \frac{\rho \omega}{D} \frac{\partial \psi}{\partial \phi}$$

and

$$\frac{Q_o}{\Delta H_o} \frac{\partial \Delta H}{\partial Q} = \frac{\phi_o}{\psi_o} \frac{\partial \psi}{\partial \phi}$$

We are interested in cushion pressure p, which is related to ΔH by

$$p = \Delta H - (1 - \eta_T) \frac{1}{2} \rho \left(\frac{Q}{A}\right)^2$$

where η_T is the total head efficiency.* This equation assumes that the plenum (cushion) velocity is negligible compared with (Q/A).

* Head loss after the fan/$\frac{1}{2}\rho(Q/A)^2$ where A is the fan duct area

$$\therefore \qquad \frac{\partial p}{\partial Q} = \frac{\partial \Delta H}{\partial Q} - (1 - \eta_T) \frac{\rho Q}{A^2}$$

$$\frac{Q_o}{p_o} \frac{\partial p}{\partial Q} = \frac{Q_o}{\left[\Delta H_o - (1 - \eta_T) \frac{1}{2} \rho \left(\frac{Q_o}{A}\right)^2\right]} \left\{ \frac{\partial \Delta H}{\partial Q} - (1 - \eta_T) \frac{\rho Q_o}{A^2} \right\}$$

$$= \frac{\left[\frac{\partial \psi}{\partial \phi} - (1 - \eta_T) \frac{\phi_o}{\pi^2}\right]}{\left[\frac{\psi_o}{\phi_o} - (1 - \eta_T) \frac{\phi_o}{\pi^2}\right]}$$

A common case (and also the lower limit on η_T) is for the diffusion process to approximate the Borda-Carnot "rapid diffusion" case for which

$$(1 - \eta_T) = \left(1 - \frac{A}{S}\right)^2, \qquad \eta_T = 2 \frac{A}{S} (1 - A/S)$$

- where A is the total fan area, and S the cushion area.

APPENDIX II

THE COMPRESSIBLE EQUATION FOR Δp

If

$$J_1 = (1 + p_o/p_\infty)^{1/\gamma} \qquad J_2 = (p_o/p_\infty)(1 + p_o/p_\infty)^{1/\gamma - 1}$$

$$\hat{p} = \Delta p/p_o \qquad \hat{V} = V/\gamma Q_o \qquad \xi = (1/Q_o)(dV/dt)$$

Then equation (11) becomes

$$1 + \frac{1}{\zeta}\hat{p} - \sqrt{1+\hat{p}} = J_1\xi + \hat{V}J_2\frac{d\hat{p}}{dt} \tag{II.1}$$

Here ξ and $\hat{V}$ are the driving terms, and may be quite general. In our present case

$$\hat{V} = \hat{V}_o - \hat{V}_1 \sin \Omega t$$

$$\therefore \quad \frac{dV}{dt} = \gamma Q_o \frac{d\hat{V}}{dt} = -\gamma Q_o \Omega \hat{V}_1 \cos \Omega t$$

$$\therefore \quad \xi = -\gamma\Omega\hat{V}_1 \cos \Omega t$$

Making these subsitutions in equation (II.1)

$$J_2(\hat{V}_o - \hat{V}_1 \sin \Omega t)\frac{d\hat{p}}{dt} + \sqrt{1+\hat{p}} - \frac{1}{\zeta}\hat{p} = 1 + J_1\gamma\Omega\hat{V}_1 \cos \Omega t \tag{II.2}$$

Several simplifications immediately come to mind. For low accelerations $\hat{p} << 1$, so

$$\sqrt{1 + \hat{p}} \approx 1 + \frac{1}{2}\hat{p}$$

The same assumption implies $\hat{V}_1 << \hat{V}_o$. Thus

$$\frac{d\hat{p}}{dt} + \frac{\left(\frac{1}{2} - \frac{1}{\zeta}\right)}{J_2\hat{V}_o}\hat{p} = \frac{J_1\gamma\Omega\hat{V}_1}{J_2\hat{V}_o}\cos \Omega t \tag{II.3}$$

or

$$\frac{d\hat{p}}{dt} + A\hat{p} = B \cos \Omega t$$

$$\therefore \quad \hat{p} = e^{-At}\, B\int e^{At} \cos \Omega t \, dt + \text{transient terms}$$

If $\theta = \Omega t$, $t = \theta/\Omega$

$$\hat{p} = e^{-A\theta/\Omega} \frac{B}{\Omega} \int e^{A\theta/\Omega} \cos\theta \, d\theta$$

$$= \left(\frac{BA}{\Omega^2 + A^2}\right)\cos\Omega t + \left(\frac{B\Omega}{\Omega^2 + A^2}\right)\sin\Omega t$$

Substituting for A and B

$$\frac{\Delta p}{p_o} = \frac{J_1\gamma\Omega\hat{V}_1\left(\frac{1}{2} - \frac{1}{\zeta}\right)\cos\Omega t + J_2J_1\gamma\Omega^2\hat{V}_o\hat{V}_1\sin\Omega t}{\Omega^2J_2{}^2\hat{V}_o{}^2 + \left(\frac{1}{2} - \frac{1}{\zeta}\right)^2} \tag{II.4}$$

$$= \frac{\Omega\hat{V}_1\gamma J_1\cos(\Omega t - \phi_2)}{\sqrt{(\Omega J_2\hat{V}_o)^2 + \left(\frac{1}{2} - \frac{1}{\zeta}\right)^2}} \tag{II.5}$$

where $$\cos\phi_2 = \frac{\Omega\hat{V}_1\gamma J_1\left(\frac{1}{2} - \frac{1}{\zeta}\right)}{\sqrt{(\Omega J_2\hat{V}_o)^2 + \left(\frac{1}{2} - \frac{1}{\zeta}\right)^2}}$$

The incompressible flow result, equation (25), may be written as

$$\frac{\Delta p}{p_o} = \Omega\gamma\hat{V}_1\left(\frac{2\zeta}{2-\zeta}\right)\cos\Omega t$$

$$= (\Omega\gamma\hat{V}_1)/\left(\frac{1}{\zeta} - \frac{1}{2}\right)\cos\Omega t \tag{II.6}$$

(Also the result of putting $d\hat{p}/dt = 0$ and $J_1 = 1$ in equation II.3).

Thus the relative amplitude

$$r = \frac{(\Delta p/p_o)_{comp}}{(\Delta p/p_o)_{incomp}} = \frac{J_1\left(\frac{1}{2} - \frac{1}{\zeta}\right)}{\sqrt{(\Omega J_2\hat{V}_o)^2 + \left(\frac{1}{2} - \frac{1}{\zeta}\right)^2}} \tag{II.7}$$

$r \to J_1$ as $\zeta \to 0$, a result we have seen already.

The phase angle can be expressed as

$$\cos \phi_2 = r\gamma\hat{V}_1\Omega = \frac{rV_1\Omega}{Q_o} \tag{II.8}$$

Generally speaking $J_1 \approx 1$, $J_2 \approx p_o/p_\infty$. Also

$$\Omega J_2\hat{V}_o \approx \frac{p_o}{p_\infty}\frac{V_o}{\gamma Q_o}\frac{2\pi U}{\lambda} \tag{II.9}$$

$$= \frac{2\pi}{\gamma}\left(\frac{H_o}{h_w}\right)\left(\frac{h_w}{\lambda}\right) U \left(\frac{p_o}{p_\infty}\right)\left(\frac{W}{p_o Q_o}\right) \tag{II.10}$$

which is similar in form to G/γ in equation (13).

Note that r is always less than unity, according to this theory, in direct contradiction of earlier results. But in fact, this follows from Equation II.3. Whenever a first order linear equation is driven by a sinusoid, its amplitude will always be less than the value obtained when the differential is neglected.

APPENDIX III

A SIMPLE CHECK ON FAN MODULATION POWER FOR 100% ATTENUATION

Suppose the cushion pressure is held constant at the trim value p_o, so that the total head rise through the fan must be

$$\Delta H = p_o + (1 - \eta_T) \frac{1}{2} \rho (Q/A)^2 \tag{III.1}$$

The instantaneous fan power requirement will therefore be

$$\eta_F P = \Delta HQ = p_o Q + (1 - \eta_T) \frac{1}{2} \rho (Q/a)^2 \tag{III.2}$$

Let

$$Q = Q_o(1 + K_w \cos \Omega t) = Q_o(1 + K_w \cos \theta) \tag{III.3}$$

Now in equilibrium,

$$\eta_F P_o = p_o Q_o + \frac{1}{2} \rho (Q_o/A)^2 Q_o (1 - \eta_T)$$

$$\therefore \quad \frac{1}{2} \rho (Q_o/A)^2 Q_o (1 - \eta_T) = \eta_F P_o - p_o Q_o$$

$$= P_o(\eta_F - \eta_L) \tag{III.4}$$

Thus the instantaneous power can be expressed as

$$\eta_F P = p_o Q_o (1 + K_w \cos \theta) + P_o(\eta_F - \eta_L)(1 + K_w \cos \theta)^3$$

or

$$\frac{P}{P_o} = \frac{\eta_L}{\eta_F} (1 + K_w \cos \theta) + (1 - \eta_L/\eta_F)(1 + K_w \cos \theta)^3 \tag{III.5}$$

$$= \frac{\eta_L}{\eta_F} (1 + K_w \cos \theta) + (1 - \eta_L/\eta_F)(1 + 3K_w \cos \theta + 3K_w^2 \cos^2\theta + K_w^3 \cos^3\theta)$$

Thus the average over one cycle is

$$\frac{P_{av}}{P_o} = \frac{\eta_L}{\eta_F} + \frac{3}{2} K_w^2 (1 - \eta_L/\eta_F) \tag{III.6}$$

which is the same as equation 48 in the main body of the report.